2 Basic concepts and equations

A plasma is a partially or completely ionized gas, which is electrically neutral as a whole, and which consists of electrons, ions and neutral atoms. Furthermore, the plasma, in the sense used in the following, is characterized by so called collective behaviour of its particles. This aspect is used to be expressed by the following conditions (for the electron-proton plasma):

a) The mean force of near interactions is much less than that of distant collective interactions of particles

$$< F_{near} > \ll < F_{dist} >,$$

b) the number of particles in the so called Debye sphere is large

$$\frac{1}{n\lambda_D^3} \ll 1,$$

where n is the plasma particle density and λ_D is the Debye length,
c) the thermal kinetic energy KE is much greater than potential energy PE

$$KE \gg PE, \quad \frac{3}{2}k_B T \gg \frac{e^2}{4\pi\epsilon_0\lambda_D},$$

where k_B is the Boltzmann constant, T the plasma temperature, e the elementary charge, and ϵ_0 the permittivity of the free space.

It means that a plasma is a sufficiently diluted and hot gas, which characteristic length L is much greater than the Debye length ($L \gg \lambda_D$).

2.1 Debye shielding

Let us assume a charge q_0 at zero point of the reference system at $\mathbf{r}_0 = 0$. The potential of this charge in free space is

$$\varphi_0(\mathbf{r}) = \frac{q_0}{4\pi\epsilon_0 \mid \mathbf{r} \mid}. \tag{1}$$

Now, let us consider a test charge q_0 surrounded by a neutral plasma (electrons with the electron density n_e and electron temperature T_e and heavy protons of the same density $n_e = n_p$). Then the potential φ can be determined from Poisson equation ($\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0}$ and $\mathbf{E} = -\nabla\varphi$, where $\mathbf{E}$ is the electric field and ρ_e is the charge density)

$$\nabla^2\varphi(\mathbf{r}) = -\frac{q_0}{\epsilon_0}\delta(\mathbf{r}) + \frac{e}{\epsilon_0} < \rho_e >, \tag{2}$$

where $\delta(\mathbf{r})$ is the delta function and the mean charge density is

$$< \rho_e >= n_e \exp\left(\frac{e\varphi}{k_B T_e}\right) - n_p. \tag{3}$$

Here for electrons the Maxwell-Boltzmann statistics ($\sim \exp(-q\varphi/(k_BT))$) is used. As mentioned above, in the plasma the kinetic energy of electrons is much greater than their potential energy, and that is why the exponential function can be expanded in a Taylor series and only two first terms can be used. Thus, the charge density in the form of $< \rho_e > \approx n_e e\varphi/(k_BT_e)$ can be put into Poisson equation which can be written as

$$-\nabla^2\varphi_0(r) + \frac{n_e e^2\varphi(r)}{\epsilon_0 k_B T_e} = \frac{q_0}{\epsilon_0}\delta(\mathbf{r}). \tag{4}$$

Solving this equation in spherical coordinates (i.e. $\nabla^2 \to \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ and making these derivatives) the solution is

$$\varphi_0(\mathbf{r}) = \frac{q_0}{4\pi\epsilon_0 r}\exp(-\frac{r}{\lambda_D}), \tag{5}$$

where

$$\lambda_D \equiv \sqrt{\left(\frac{\epsilon_0 k_B T}{n_e e^2}\right)}, \tag{6}$$

is the Debye radius. As seen the electric field ($\mathbf{E} = -\nabla\varphi$) at distances $r \geq \lambda_D$ is effectively shielded.

2.2 Plasma oscillations

In a quasi-neutral electron-ion plasma a relative displacement of electrons in comparison with ions causes the electric field:

$$\frac{\partial E}{\partial x} = \frac{n_e e}{\epsilon_0} \Rightarrow E = \frac{n_e e\xi}{\epsilon_0}, \tag{7}$$

where x is the spatial coordinate and ξ the displacement. Then the Newton force equation gives the equation for oscillations of electrons

$$m_e\frac{d^2}{dt^2}\xi = -eE = -\frac{n_e}{\epsilon_0}e^2\xi, \tag{8}$$

(where m_e is the electron mass) with the characteristic frequency called electron plasma frequency

$$\omega_{pe}^2 = \frac{n_e e^2}{\epsilon_0 m_e}. \tag{9}$$

Similarly, we can define the proton plasma frequency as

$$\omega_{pp}^2 = \frac{n_p e^2}{\epsilon_0 m_p}, \tag{10}$$

where m_p is the proton mass, and so on, e.g. for charged dust particles (dust plasma).

2.3 Equations describing plasma processes

2.3.1 Kinetic description

A plasma is a system of an enormous amount of particles, therefore the statistical approach is used for its description. Generally, the plasma can be described by the N-particle distribution function $f_N(z_1, z_2, z_3, ..., z_N, t)$, where $z_1 = (\mathbf{r}_1, \mathbf{p}_1)$, $z_2 = (\mathbf{r}_2, \mathbf{p}_2)$ mean positions $(\mathbf{r}_1, \mathbf{r}_2)$ and impulses $(\mathbf{p}_1, \mathbf{p}_2)$ of particle 1, 2, and so on. Due to the Liouville theorem, the function f_N fulfils the continuity equation in the 6N dimensional phase space

$$\frac{\partial f_N}{\partial t} + \sum_{l=1}^{N} \frac{\partial}{\partial z_l}(\dot{z}_l f_N) = 0, \tag{11}$$

where $\dot{z}_l$ is the time derivative of z_l. Using now the Hamilton equations

$$\dot{\mathbf{r}}_l = \frac{\partial H}{\partial \mathbf{p}_l}, \tag{12}$$

$$\dot{\mathbf{p}}_l = -\frac{\partial H}{\partial \mathbf{r}_l}, \tag{13}$$

(where H is the Hamiltonian of the plasma) we can write

$$\frac{\partial f_N}{\partial t} + [f_N, H] = 0, \tag{14}$$

where [...] is the Poisson bracket

$$[f_N, H] = \sum_{l=1}^{N} \left(\frac{\partial H}{\partial \mathbf{p}_l} \frac{\partial f_N}{\partial \mathbf{r}_l} - \frac{\partial H}{\partial \mathbf{r}_l} \frac{\partial f_N}{\partial \mathbf{p}_l} \right). \tag{15}$$

As described in the book of Achiezer et al. (1974) the equation (14) can be transformed into a chain of equations by integration over part of the variables. The first equation connects the evolution of the one-particle distribution function with the two-particle distribution function, second equation connects the two-particle distribution function with the three-particle distribution function, and so on (Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy). Considering in this chain of equations only the first one (for the one-particle distribution function) and neglecting effects of the two-particle distribution function, plasma can be described by a distribution function $f(\mathbf{r}, \mathbf{v}, t)$. The distribution gives the number of particles which are present in a unit volume of the 6-dimensional phase space located in the spatial coordinate $\mathbf{r}$, and the velocity coordinate $\mathbf{v} = \mathbf{p}/m$ (in the classical case) at time t. The distribution function is thus a solution of the so called **Boltzmann equation**

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} = \left(\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} \right)_{coll}, \tag{16}$$

where m is the particle mass, $\mathbf{F}$ is the general force, and in our case usually in the form

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{17}$$

where q is the particle charge, and $\mathbf{E}$ and $\mathbf{B}$ are the electric field and magnetic induction.

The term on the right side of the equation (16) expresses effects of particle collisions.

Because plasma can have many different components (electrons, protons, neutrals, ions of different chemical elements), the Boltzmann equation should be solved for every single plasma component and interactions between components should appear in collisional terms on the right side of the individual Boltzmann equations. But, for many tasks some simplifications are made, and, e.g., only the Boltzmann equation for electrons is solved.

Furthermore, if the collisional term is very low (e.g., if the plasma frequency ω_{pe} is much greater than the collision frequency ν_c; $\omega_{pe} \gg \nu_c$) then such a plasma is called collisionless and for its description the **Vlasov equation** is used

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} = 0. \tag{18}$$

For a full set of equations describing a plasma behaviour the **Maxwell equations** need to be added

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad & \nabla \cdot \mathbf{E} &= \frac{\rho_e}{\varepsilon_0}, \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \qquad & \nabla \cdot \mathbf{B} &= 0, \end{aligned} \tag{19}$$

where $\mathbf{j}$ is the electric current density and ρ_e is the charge density, which can be expressed as follows

$$\mathbf{j} = \sum_\alpha e_\alpha \int \mathbf{v} f_\alpha d^3 v, \tag{20}$$

$$\rho_e = \sum_\alpha e_\alpha \int f_\alpha d^3 v, \tag{21}$$

where the index α corresponds to individual plasma components. For more details in the kinetic theory, see also Hadrava (2013).

2.3.2 Fokker-Planck equation

If the particle collisions are dominant then an evolution of the particle distribution function is described by the **Fokker-Planck equation**. Let $P(\mathbf{v}, \Delta\mathbf{v})$ is the probability that a test particle changes its velocity $\mathbf{v}$ to $\mathbf{v} + \Delta v$ in the time interval Δt. If the particle number is conserved then the velocity distribution at time t can be expressed as

$$f(\mathbf{v}, t) = \int f(\mathbf{v} - \Delta v, t - \Delta t) P(\mathbf{v} - \Delta v, \Delta v) d^3 \Delta v. \tag{22}$$

For small-angle deflections $|\Delta v| \ll |v|$, the product fP in Eq. (22) can be expanded into a Taylor series,

$$f(\mathbf{v},t) = \int (fP - \Delta t[\frac{\partial f}{\partial t}]P - \Delta v_i[\frac{\partial}{\partial v_i} fP] + \frac{1}{2}\Delta v_i \Delta v_j [\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j} fP] + ...)d^3\Delta v. \quad (23)$$

We introduced the Einstein convention that the sums over the indices i and j have to be used if they appear together in the numerator and denominator, or as subscripts and superscripts. Because the probability that some transition takes place is unity, P is normalized to

$$\int P d^3\Delta v = 1. \quad (24)$$

We express the average velocity change per time interval Δt as:

$$\int \Delta \mathbf{v} P d^3 \Delta v \equiv < \Delta \mathbf{v} >, \quad (25)$$

$$\int \Delta v_i \Delta v_j P d^3 \Delta v \equiv < \Delta v_i \Delta v_j > . \quad (26)$$

We exchange integration and differentiation. Then the integral in Eq. (23) can be evaluated. The first term in the integrand cancels with the left hand side of the equation. The other terms form **Fokker-Planck equation**:

$$\left(\frac{\partial f(\mathbf{v},t)}{\partial t}\right)_{coll} = \frac{\partial^2}{\partial v_i \partial v_j}\left(f\frac{< \Delta v_i \Delta v_j >}{2\Delta t}\right) - \frac{\partial}{\partial v_i}\left(f\frac{< \Delta v_i >}{\Delta t}\right). \quad (27)$$

Due to a property of inverse-square law particles having multiple collisions we can neglect the higher-order terms in the expansion (23). Equation (27) shows that the motion of particles in velocity space can be described as a diffusion process. Its right hand side describes the temporal change of a distribution of test particles by multiple, small-angle collision processes. It corresponds to the right hand side of the Boltzmann equation (16). The first term in the right side of Eq. (27) represents the three-dimensional diffusion of the test particle in velocity space; the second term is a friction, slowing down the test particle and moving it radially toward the origin of velocity space in the local rest frame of the colliding particles.

2.4 Magnetohydrodynamic description

For many tasks in astrophysical plasmas the kinetic approach is too complex, in reality we do not need to know distribution functions of plasma particles. In these cases the description using the macroscopic quantities as, e.g., the mean plasma velocity and so on is sufficient. Mathematically it means that the integration of kinetic equations in velocity space is justifiable. Thus, the equations with the macroscopic quantities (called the magnetohydrodynamic equations, MHD equations for short) can be obtained as the moments of the Boltzmann equation $[BKE]$:

$$\int [BKE] d^3 v, \quad (28)$$