A. Measures and Measurable Functions

1. The Lebesgue Measure

In the history, people were engaged in the problem of measuring lenghts, areas and volumes. In mathematical formulation the task was, for a given set A, to determine its size ("measure") λA. It was required that the volume of a cube or the area of a rectangle or a circle should agree with the well-known formulae. It was also clear by intuition that this measure should be positive and additive, i.e. it should satisfy the equality

$$\lambda \bigcup A_j = \sum \lambda A_j$$

provided $\{A_j\}$ is a finite disjoint collection of sets. For a succesful development of the theory a further condition was imposed: The above equality was claimed to hold even for countable disjoint collections of sets. Moreover, the effort was paid to assign a measure to as many sets as possible.

Now, we are going to show how to proceed on the real line. The same approach will be used later in the Euclidean space $\mathbf{R}^n$ where the proofs will be given.

1.1. Outer Lebesgue Measure. For an arbitrary set $A \subset \mathbf{R}$, define

$$\lambda^* A := \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \colon \bigcup_{i=1}^{\infty}(a_i, b_i) \supset A\}.$$

The value $\lambda^* A$ (which can also be $+\infty$) is called the *outer Lebesgue measure* of a set A.

1.2. Properties of the Outer Lebesgue Measure. One can see immediately that $\lambda^* A \leq \lambda^* B$ if $A \subset B$ and that the measure of a singleton is 0, and without much effort it becomes clear that $\lambda^* I$ is the length of I in case of I interval of any type (see Exercise 1.6). Then it is relatively easy to prove that the outer Lebesgue measure is *translation invariant*: If $A \subset \mathbf{R}$ and $x \in \mathbf{R}$, then $\lambda^* A = \lambda^*(x + A)$. Another important property is the *σ-subadditivity*:

$$\lambda^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \lambda^* A_j.$$

In mathematical terminology, the prefix σ usually relates to countable unions and δ to countable intersections.

The question of whether λ^* is an additive set function has a negative answer: There are disjoint sets A, B with

$$\lambda^*(A \cup B) < \lambda^* A + \lambda^* B$$

(cf. 1.8), and we need to find a family of sets (as large as possible) on which the measure λ^* is additive. This task will be solved later in Chapter 4 in a much more general case. Now we just briefly indicate one of its possible solutions in case of the Lebesgue measure.

1.3. Lebesgue Measurable Sets. Let A be a subset of a bounded interval I. Defining the "inner measure" $\lambda_* A = \lambda^* I - \lambda^*(I - A)$, it is natural to investigate the collection of sets for which $\lambda_* A = \lambda^* A$ (cf. Exercise 1.7). This leads to the following definition.

We say that a set $A \subset \mathbf{R}$ is (Lebesgue) *measurable* if $\lambda^* I = \lambda^*(A \cap I) + \lambda^*(I \setminus A)$ for every bounded interval $I \subset \mathbf{R}$. The collection of all measurable sets on $\mathbf{R}$ will be denoted by $\mathfrak{M}$. Not every set is measurable as will be seen in 1.8. The set function $M \mapsto \lambda^* M$, $M \in \mathfrak{M}$ is denoted by λ and called the *Lebesgue measure*. Thus, on measurable sets, the set functions λ^* and λ coincide but for nonmeasurable ones only λ^* is defined.

Another important property of the measure λ is contained in the following theorem which is now presented without proof.

1.4. Theorem. *(a) If $M_1, M_2, \dots$ are elements of $\mathfrak{M}$, then also $M_1 \setminus M_2$, $\bigcap M_n$ and $\bigcup M_n$ are elements of $\mathfrak{M}$. If, in addition, the sets M_n are pairwise disjoint, then*

$$\lambda\Big(\bigcup_n M_n\Big) = \sum_n \lambda M_n.$$

(b) Intervals of any type are in $\mathfrak{M}$.

1.5. Remark. The ingenuity of Lebesgue's approach to the measure consists in considering the *countable* covers of a set A with intervals. If in the definition of $\lambda^* A$ we consider only *finite* covers, we get the notion of so-called *Jordan-Peano content*. In modern analysis this notion is far from being as important as the Lebesgue measure.

1.6. Exercise. If $I \subset \mathbf{R}$ is an interval (of any type), show that $\lambda^* I$ is its length.

Hint. It is sufficient to consider the case $I = [a, b]$. Clearly $\lambda^*[a, b] \le b - a$ (since $[a, b] \subset (a - \varepsilon, b + \varepsilon)$). Suppose $\bigcup\limits_{i=1}^{\infty} (a_i, b_i) \supset [a, b]$. A compactness argument yields the existence of an index n satisfying $\bigcup\limits_{i=1}^{n} (a_i, b_i) \supset [a, b]$. Using induction (with respect to n) it can be shown that $b - a \le \sum\limits_{i=1}^{n} (b_i - a_i)$.

1.7. Exercise. For every bounded set $A \subset \mathbf{R}$, define

$$\lambda_* A := \lambda I - \lambda^*(I \setminus A)$$

where I is a bounded interval containing A. Show that:

(a) the value of $\lambda_* A$ does not depend on the choice of I;

(b) a bounded set $A \subset \mathbf{R}$ is measurable if and only if $\lambda^* A = \lambda_* A$;

(c) a set $M \subset \mathbf{R}$ is measurable if and only if its intersection with each bounded interval is measurable.

In the next part of this chapter we introduce some significant sets on the real line.

1.8. A Nonmeasurable Set. Now we prove the existence of a nonmeasurable subset of $\mathbf{R}$ and consequently prove that the outer Lebesgue measure cannot be additive.

Set $x \sim y$ if $x - y$ is a rational number. It is easy to see that $\sim$ is an equivalence relation on $\mathbf{R}$. Therefore $\mathbf{R}$ splits into an uncountable collection $\mathscr{V}$ of pairwise disjoint classes. A set V belongs to this collection $\mathscr{V}$ if and only if $V = x + \mathbf{Q}$ for some $x \in \mathbf{R}$. By the axiom of choice, there exists a set $E \subset (0,1)$ that shares exactly one point with each set $V \in \mathscr{V}$. We show that E is not in $\mathfrak{M}$.

Let $\{q_n\}$ be a sequence containing all rational numbers from the interval $(-1,+1)$. It is not very difficult to show that the sets $E_n := q_n + E$ are pairwise disjoint and that

$$(0,1) \subset \bigcup_n E_n \subset (-1,2).$$

Assuming that $E \in \mathfrak{M}$, then also $E_n \in \mathfrak{M}$ and Theorem 1.4 gives $\lambda \bigcup\limits_n E_n = \sum\limits_n \lambda E_n$. Distinguishing two cases $\lambda E = 0$ and $\lambda E > 0$ we easily obtain the contradiction.

1.9. Remarks. 1. The proof of the existence of a nonmeasurable set is not a constructive one (it uses the axiom of choice for an uncountable collection of sets). We return to the topic of nonmeasurable sets in Notes 1.22.

2. By a simple argument, an even stronger proposition can be proved: Any measurable set $M \subset \mathbf{R}$ of a positive measure contains a nonmeasurable subset. It is sufficient to realize that $M = \bigcup_{q \in \mathbf{Q}} M \cap (E+q)$ where E is the nonmeasurable set from 1.8 and that any measurable subset of E is of zero (Lebesgue) measure.

3. Van Vleck [1908] "constructed" a set $E \subset [0,1]$ for which $\lambda^* E = 1$ and $\lambda_* E = 0$.

1.10. Exercise. Show that every countable set S is of measure zero.

Hint. Consider covers $\bigcup\limits_{j=1}^{\infty} (r_j - \varepsilon 2^{-j}, r_j + \varepsilon 2^{-j})$ where $\{r_j\}$ is a sequence of all elements of the set S. The assertion also follows from Theorem 1.4 if you realize that singletons have measure zero.

1.11. Examples of Sets of Measure Zero. (a) The set $\mathbf{Q}$ of all rational numbers is countable, thus by Exercise 1.10 it has Lebesgue measure zero.

(b) It can be seen from the hint to the exercise that for every $k \in \mathbf{N}$ there is an open set G_k such that $\mathbf{Q} \subset G_k$ and $\lambda^* G_k \le 1/k$. The set $\bigcap\limits_{k=1}^{\infty} G_k$ has also Lebesgue measure zero, it is dense and uncountable (even residual).

1.12. Cantor Ternary Set. Consider the sequence $\{\mathscr{K}_n\}$ of finite collections of intervals defined in the following way: $\mathscr{K}_0 = \{[0,1]\}$, $\mathscr{K}_1 = \{[0,\frac{1}{3}],[\frac{2}{3},1]\}$. In each step we construct $\mathscr{K}_n$ from $\mathscr{K}_{n-1}$ as the collection of all closed intervals which are the left or right third of an interval from the collection $\mathscr{K}_{n-1}$ (the middle thirds are omitted). Then $\mathscr{K}_n$ is a collection of 2^n disjoint closed intervals, each of them of length 3^{-n}. Let K_n denote the union of the collection $\mathscr{K}_n$. The *Cantor ternary set*[1] C is defined as $\bigcap\limits_n K_n$. It is not difficult to verify that C consists precisely of points of the form $\sum\limits_{i=1}^{\infty} a_i 3^{-i}$ where each a_i is 0 or 2. Roughly speaking, in the Cantor set there are exactly those points of the interval $[0,1]$ whose ternary expansions do not contain the digit 1. The Cantor set has the following properties:

(a) C is a compact set without isolated points;

(b) C is a nowhere dense (and totally disconnected) set;

(c) C is an uncountable set;

(d) the Lebesgue measure of C is zero.

1.13 Discontinua of a Positive Measure. If we construct a set $D \subset [0,1]$ like the Cantor set except that we always omit intervals of length $\varepsilon 3^{-n}$ where $\varepsilon \in (0,1)$ (note that their centres are not the same as those in the construction of the Cantor set), we get a closed nowhere dense set, for which $\lambda D = 1 - \varepsilon$. Sets having this property are called the *discontinua of a positive measure*. Another construction: If G is an open subset of the interval $(0,1)$, containing all rational points of this interval and $\lambda G = \varepsilon < 1$ then $[0,1] \setminus G$ is a discontinuum of measure $1 - \varepsilon$.

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[1] sometimes also called the *Cantor discontinuum*

1.14. Exercise. Prove that there exists a non-Borel subset of the Cantor set and realize that this set is Lebesgue measurable.

Hint. The cardinality argument shows that the set of all Borel subsets of the Cantor set has cardinality of the continuum while the set of all its subsets has greater cardinality.

Instead of this, the following idea can be used. Define

$$\kappa(t) := \inf\{x \in [0,1] : f(x) = t\}$$

where f is the Cantor singular function from 23.1. Show that κ is increasing on the interval $[0,1]$, and therefore it is a Borel function. Suppose E is a nonmeasurable subset of $[0,1]$, $B := \kappa(E)$. Then B (as a subset of the Cantor set) is a measurable set. But since $\kappa^{-1}(B) = E$ (and κ is a Borel function), B cannot be a Borel set.

1.15. Lebesgue Measure on $\mathbf{R}^n$. In the same way as for $\mathbf{R}$, we introduce the Lebesgue measure on $\mathbf{R}^n$. Recall that by an interval in $\mathbf{R}^n$ we understand an arbitrary Cartesian product of n one-dimensional intervals. If $I := (a_1, b_1) \times \cdots \times (a_n, b_n)$ is an open interval, we define its *volume* as

$$\operatorname{vol} I = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n).$$

In the same way we define $\operatorname{vol} I$ for intervals of other types. Given an arbitrary set $A \subset \mathbf{R}^n$, define the *outer Lebesgue measure* of A as the quantity

$$\lambda^* A = \inf\{\sum_{k=1}^{\infty} \operatorname{vol} I_k \colon \bigcup_{k=1}^{\infty} I_k \supset A,\ I_k \text{ is an open interval}\}.$$

We say that a set $A \subset \mathbf{R}^n$ is *measurable* if $\lambda^* T = \lambda^*(A \cap T) + \lambda^*(T \setminus A)$ for every set $T \subset \mathbf{R}^n$. (By analogy with the one-dimensional case we should require this equality to hold just for bounded intervals T. We have chosen the present definition in order to apply the general approach of Chapter 4. Soon we show that there is no difference between these two definitions.) The symbol $\mathfrak{M}$ again denotes the collection of all measurable subsets of $\mathbf{R}^n$. For $M \in \mathfrak{M}$ we denote by $\lambda M := \lambda^* M$ the n-dimensional *Lebesgue measure* of a set M.

1.16. Theorem. *If $\{A_j\}$ is a sequence of (arbitrary) sets of $\mathbf{R}^n$, then*

$$\lambda^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \le \sum_{j=1}^{\infty} \lambda^* A_j.$$

Proof. The assertion follows from Theorem 4.3. ∎

1.17. Theorem. *If M_1, $M_2, \ldots$ are elements of $\mathfrak{M}$, then also $M_1 \setminus M_2$, $\bigcap M_n$ and $\bigcup M_n$ are elements of $\mathfrak{M}$. If, in addition, the sets M_n are pairwise disjoint, then*

$$\lambda\Big(\bigcup_n M_n\Big) = \sum_n \lambda M_n.$$

Proof. The assertion follows from general Theorem 4.5. ∎

Compare the following theorem with Exercise 1.6.

1.18. Theorem. *If $I \subset \mathbf{R}^n$ is a bounded interval, $I \subset \bigcup_j Q_j$ where $\{Q_j\}$ is a sequence of open intervals, then*

$$\operatorname{vol} I \le \sum_j \operatorname{vol} Q_j.$$

Thus the n-dimensional Lebesgue measure $\lambda^* I$ is equal to the volume $\operatorname{vol} I$.

Proof. Suppose J is a compact interval contained in I. There exists a p such that the intervals $\{Q_1, \dots, Q_p\}$ cover J. The interval J can be now divided into a finite number of non-overlapping n-dimensional intervals $\{J_i\}$ (distinct elements of $\{J_i\}$ have disjoint interiors) in such a way that the interior of each interval J_i is contained in some of the intervals Q_j. Then

$$\operatorname{vol} J = \sum_i \operatorname{vol} J_i \le \sum_{j=1}^{p} \operatorname{vol} Q_j \le \sum_{j=1}^{\infty} \operatorname{vol} Q_j.$$

Since the difference $\operatorname{vol} I - \operatorname{vol} J$ can be arbitrarily small, the assertion follows. ■

1.19. Theorem. (a) *Any open subset of $\mathbf{R}^n$ is measurable.*

(b) *If $\lambda^* A = 0$, then A is measurable.*

Proof. The proof of part (b) is obvious; we will prove (a). First we prove that each interval H which is a halfspace (e.g. of the form $(-\infty, c) \times \mathbf{R}^{n-1}$) is measurable. Choose a "test" set T, $\lambda^* T < \infty$, and $\varepsilon > 0$. There exist open intervals $\{Q_j\}$ with

$$\bigcup_j Q_j \supset T \text{ and } \sum_j \operatorname{vol} I_j < \lambda^* T + \varepsilon.$$

Now find open intervals I_j and J_j such that

$$I_j \cup J_j = Q_j, \quad Q_j \cap H \subset I_j,\ Q_j \setminus H \subset J_j \text{ and } \lambda^* I_j + \lambda^* J_j < \lambda^* Q_j + \varepsilon 2^{-j}.$$

Then

$$\lambda^*(T \cap I) + \lambda^*(T \setminus I) \le \sum_j \operatorname{vol} I_j + \sum_j \operatorname{vol} J_j \le \lambda^* T + \varepsilon.$$

We proved the measurability of all intervals H of the form of a halfspace. Now, each open set can be expressed as a countable union of intervals and each interval is a finite intersection of intervals which are halfspaces. ■

1.20. Theorem. *If $A \subset \mathbf{R}^n$, then*

$$\lambda^* A = \inf \{\lambda G \colon G \text{ open}, G \supset A\}.$$

Proof. One inequality follows from the monotonicity of λ^*. Now if $\lambda^* A < \infty$ and $\varepsilon > 0$, then there exist open intervals $I_j \subset \mathbf{R}^n$ such that

$$A \subset \bigcup_j I_j \quad \text{and} \quad \lambda \bigcup_j I_j \le \sum_j \operatorname{vol} I_j < \lambda^* A + \varepsilon.$$

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The reader should compare the following theorem and Exercise 15.19.

1.21. Theorem. *Given a set $M \subset \mathbf{R}^n$, the following are equivalent:*

(i) *M is measurable;*
(ii) *for every bounded interval I, $\lambda^* I = \lambda^*(I \cap M) + \lambda^*(I \setminus M)$;*
(iii) *for every $\varepsilon > 0$ there exists an open set $G \supset M$ with $\lambda^*(G \setminus M) < \varepsilon$;*
(iv) *there exists a G_δ-set $D \supset M$ such that $\lambda^*(D \setminus M) = 0$;*
(v) *there exist an F_σ-set B_i and a G_δ-set B_e such that $B_i \subset M \subset B_e$ and $\lambda^*(B_e \setminus B_i) = 0$.*

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial. Assuming (ii), fix $\varepsilon > 0$ and denote $I_k = (-k, k)^n$. By Theorem 1.20 we can find open sets G_k and H_k such that $I_k \cap M \subset G_k$, $I_k \setminus M \subset H_k$, $\lambda G_k \le \lambda^*(I_k \cap M) + 2^{-k}\varepsilon$ and $\lambda H_k \le \lambda^*(I_k \setminus M) + 2^{-k}\varepsilon$. We can assume that G_k and H_k are subsets of I_k. Then we have $G_k \setminus M \subset G_k \cap H_k$. Using (ii) and the measurability of open sets we obtain

$$\lambda I_k + \lambda(G_k \cap H_k) = \lambda G_k + \lambda H_k \le \lambda^*(I_k \cap M) + \lambda^*(I_k \setminus M) + 2^{-k+1}\varepsilon \le \lambda I_k + 2^{-k+1}\varepsilon.$$

Set $G = \bigcup_k G_k$. Then

$$\lambda^*(G \setminus M) \le \sum_{k=1}^{\infty} \lambda(G_k \cap H_k) \le 2\varepsilon$$

so that (iii) holds. That (iii) implies (iv) is evident. It is not very difficult to prove the implication (iv) $\Longrightarrow$ (v). If M satisfies (v), then $M = B_i \cup (M \setminus B_i)$ where the sets B_i and $M \setminus B_i$ are measurable by Theorem 1.19 (each one for a different reason), so that (v) $\Longrightarrow$ (i). ∎

1.22. Notes. Originally, H. Lebesgue defined the outer measure on the real line using countable covers formed by intervals, exactly as explained in the text. He defined measurability as in Exercise 1.7.

At the end of the last century, various attempts to define the length or area of geometrical figures appear; in the works of G. Peano [1887] and C. Jordan [1892] even the "measures" of more complicated sets are considered.

The existence of a Lebesgue nonmeasurable set is very closely connected to the axiom of choice (for uncountable collections of sets) and the assertion that such sets exist was first proved by G. Vitali [*1905]. Solovay's result [1970] says that there exist models of the set theory (of course not satisfying the axiom of choice) in which every subset of real numbers is Lebesgue measurable. The existence of a nonmeasurable set can be proved (assuming various set conditions) in other ways as well. Constructions of Bernstein's sets (still assuming the axiom of choice) as examples of nonmeasurable sets are also interesting. Another construction of a nonmeasurable set (the axiom of choice again) based on results of the graph theory comes from R. Thomas [1985]. Using nonstandard methods, it is possible to prove the existence of a nonmeasurable set assuming the existence of ultrafilters (a weaker form of the axiom of choice; cf. M. Davis [*1977]). Recently, M. Foreman and F. Wehrung [1991] proved that the existence of a nonmeasurable set follows from the Hahn-Banach Theorem (which is again a weaker assumption than the axiom of choice).

Let us note that the Lebesgue measure can be extended to a "translation invariant" measure defined on a wider σ-algebra than is the collection of all Lebesgue measurable sets. The construction can be found e.g. in S. Kakutani and J.C. Oxtoby [1950]. However, the Lebesgue measure cannot be extended in a reasonable way to the collection of all subsets of $\mathbf{R}^n$.

It is interesting that in $\mathbf{R}$ or $\mathbf{R}^2$ there exist finitely additive extensions of the Lebesgue measure to the collection of all subsets which can also be invariant with respect to translations