Chapter 48

One-loop vacuum polarization in detail

Renormalized UV finite parts of the contributions of the one-loop QED diagrams, which represent an important portion of the radiative corrections to scattering amplitudes are, in general, quite complicated functions of external momenta. For instance, the expression for the vertex correction $\overline{\Gamma}_\mu(p', p)$ contains also higher transcendental functions like the dilogarithm (Spence's function). On the other hand, the vacuum polarization form factor $\overline{\Pi}(q^2)$ is relatively simple and can be expressed fully in terms of elementary functions. A detailed description of this quantity is the main subject of this chapter.

Let us start with the regularized expressions for $\Pi(q^2)$ that we have obtained in chapters 39 and 40. Including also the coupling factor $e^2 = 4\pi\alpha$ in the formulae (39.19), (40.24), we have

$$\Pi^{\mathrm{DR}}(q^2) = \frac{e^2}{2\pi^2}\left[\frac{1}{6}\left(\frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln\frac{m^2}{\mu^2}\right) - \int_0^1 \mathrm{d}x\, x(1-x)\, \ln\frac{m^2 - x(1-x)q^2}{m^2}\right] \tag{48.1}$$

in the dimensional regularization and

$$\Pi^{\mathrm{PV}}(q^2) = \frac{e^2}{2\pi^2}\left[\frac{1}{6}\ln\frac{M^2}{m^2} - \int_0^1 \mathrm{d}x\, x(1-x)\ln\frac{m^2 - x(1-x)q^2}{m^2}\right] \tag{48.2}$$

in the Pauli–Villars scheme.

Thus, we see that $\overline{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$ does not depend on the regularization scheme. One has, using $4\pi\alpha$ instead of e^2,

$$\overline{\Pi}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\, \ln\frac{m^2 - x(1-x)q^2}{m^2}\,. \tag{48.3}$$

A brief inspection of the integrand in the last expression reveals that one should distinguish three regions of the q^2 values, namely

$$\begin{aligned} &\text{I)} && q^2 < 0\,, \\ &\text{II)} && 0 < q^2 \le 4m^2\,, \\ &\text{III)} && q^2 > 4m^2\,, \end{aligned} \tag{48.4}$$

with regard to the distinct properties of the quadratic function

$$C(x) = m^2 - x(1-x)q^2 \tag{48.5}$$

in these kinematical areas. Indeed, for $q^2 < 0$ the function $C(x)$ is positive for any $x \in (0, 1)$ and has real zeroes outside the interval $(0, 1)$, namely

$$x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{4m^2}{|q^2|}}\right). \tag{48.6}$$

In the region II, $C(x)$ is positive for any $x \in (0, 1)$ and has no real roots whatsoever. The region III is, in a sense, the most interesting case. The function $C(x)$ then has real roots inside the interval $(0, 1)$, namely

$$x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4m^2}{|q^2|}}\right), \tag{48.7}$$

and thus it changes sign for x between 0 and 1. In particular, $C(x) < 0$ for $x \in (x_-, x_+)$ and $C(x) > 0$ for $x \in (0, x_-) \cup (x_+, 1)$. However, the negative value of $C(x)$ means that the logarithm in the integrand in (48.3) has a **non-trivial imaginary part** for $x \in (x_-, x_+)$. Thus, one may expect that the function $\overline{\Pi}(q^2)$ will be purely real for $q^2 \in (-\infty, 4m^2)$ and complex for $q^2 > 4m^2$!

The evaluation of real parts of the integral in (48.3) in the regions I, II, III is elementary, but somewhat tedious. It is clear that it can be carried out by means of partial integration, which results in integrating a rational function. We leave it to a hard-working reader as an exercise in the elementary calculus, and here we only summarize the relevant results.

I) For $q^2 < 0$, one gets

$$\overline{\Pi}(q^2) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} + \left(1 - \frac{2m^2}{|q^2|}\right)\left(\sqrt{1 + \frac{4m^2}{|q^2|}}\,\ln\frac{\sqrt{1 + \frac{4m^2}{|q^2|}} - 1}{\sqrt{1 + \frac{4m^2}{|q^2|}} + 1} + 2\right)\right]. \tag{48.8}$$

II) For $0 \le q^2 \le 4m^2$,

$$\overline{\Pi}(q^2) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} - 2\left(1 + \frac{2m^2}{q^2}\right)\left(\sqrt{\frac{4m^2}{q^2} - 1}\,\arctan\frac{1}{\sqrt{\frac{4m^2}{q^2} - 1}} - 1\right)\right]. \tag{48.9}$$

III) For $q^2 > 4m^2$,

$$\operatorname{Re}\overline{\Pi}(q^2) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} + \left(1 + \frac{2m^2}{q^2}\right)\left(\sqrt{1 - \frac{4m^2}{q^2}}\,\ln\frac{1 - \sqrt{1 - \frac{4m^2}{q^2}}}{1 + \sqrt{1 - \frac{4m^2}{q^2}}} + 2\right)\right]. \tag{48.10}$$

The evaluation of the imaginary part of $\overline{\Pi}(q^2)$ for $q^2 > 4m^2$ is quite simple, so it is worth doing it here explicitly. To begin with, let us recall what is the origin of the function $C(x)$ in the argument of the logarithm in (48.3). Going back to the expression (38.19) for C, one cannot overlook the remark that m^2 is to be understood as $m^2 - i\epsilon$, where $\epsilon > 0$ is an infinitesimal constant, ubiquitous in Feynman propagators. Then, if the real part of $C(x)$ is negative, the imaginary part of the logarithm is equal $-i\pi$. An explanatory comment is perhaps in order here: Note that the logarithm of complex variable $z = |z|e^{i\varphi}$ is $\ln z = \ln|z| + i\varphi$ and it has the branch cut on the real axis, extending from $-\infty$ to 0; with the specification of m^2 as $m^2 - i\epsilon$ in mind, we are on the lower side of the cut, where $\varphi = -\pi$.

Thus, the evaluation of $\operatorname{Im}\overline{\Pi}(q^2)$ is easy. We have

$$\overline{\Pi}(q^2) = \operatorname{Re}\overline{\Pi}(q^2) + i\operatorname{Im}\overline{\Pi}(q^2)\,,$$

where

$$i\operatorname{Im}\overline{\Pi}(q^2) = -\frac{2\alpha}{\pi}\int_{x_-}^{x_+} \mathrm{d}x\; x(1-x)(-i\pi)\,,$$

i.e.

$$\operatorname{Im}\overline{\Pi}(q^2) = 2\alpha\int_{x_-}^{x_+} \mathrm{d}x\; x(1-x)\,, \tag{48.11}$$

with $x_\pm$ being given by (48.7) as the solution of the quadratic equation

$$x^2 - x + \frac{m^2}{q^2} = 0\,. \tag{48.12}$$

So, from (48.11) we have

$$\operatorname{Im}\overline{\Pi}(q^2) = 2\alpha\left[\frac{1}{2}(x_+^2 - x_-^2) - \frac{1}{3}(x_+^3 - x_-^3)\right]. \tag{48.13}$$

Working out the last expression is a refreshing exercise in high school maths. Indeed, one may utilize the elementary identity

$$x_+^3 - x_-^3 = (x_+ - x_-)(x_+^2 + x_-x_+ + x_-^2)\,,$$

as well as the properties of roots of the quadratic equation (48.12), such as

$$x_+ + x_- = 1\,,\quad x_-x_+ = \frac{m^2}{q^2}\,,\quad x_+ - x_- = \sqrt{1-\frac{4m^2}{q^2}}\,,$$
$$x_+^2 + x_-^2 = x_+ + x_- - \frac{2m^2}{q^2} = 1 - \frac{2m^2}{q^2}\,.$$

For (48.13) one then gets, after a simple manipulation,

$$\operatorname{Im}\overline{\Pi}(q^2) = \frac{\alpha}{3}\left(1+\frac{2m^2}{q^2}\right)\sqrt{1-\frac{4m^2}{q^2}}\,. \tag{48.14}$$

The last result is in fact highly remarkable. If we denote, provisionally, $q^2 = M^2$ and come back to $e^2 = 4\pi\alpha$, the formula (48.14) reads

$$\operatorname{Im}\overline{\Pi}(q^2 = M^2) = \frac{e^2}{12\pi}\left(1+\frac{2m^2}{M^2}\right)\sqrt{1-\frac{4m^2}{M^2}}\,. \tag{48.15}$$

Now it turns out that

$$\operatorname{Im}\overline{\Pi}(q^2 = M^2) = \frac{\Gamma}{M}\,, \tag{48.16}$$

where Γ is the rate of the decay of massive vector boson ("massive photon" with mass M) into a pair of fermions, with all particles unpolarized. To appreciate this, the reader is urged to