Appendix F

Electromagnetic form factors of electron

In this appendix, we discuss briefly the general formula (50.15). As we have already mentioned before, at the tree level it is valid automatically, with $F_1(q^2) = 1$, $F_2(q^2) = 0$. Now the question is what one gets in higher orders for the matrix product

$$\mathscr{M}_\mu = \bar{u}(p')\Gamma_\mu(p', p)u(p)\,, \tag{F.1}$$

where $\Gamma_\mu(p', p)$ denotes the vertex function represented by Feynman diagrams with two external electron lines and one external photon line (pictorially, see Fig. F.1). Needless to say, p and p'

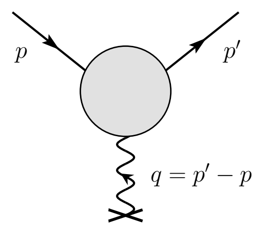

Fig. F.1: Schematic depiction of the QED vertex function that embodies the electron magnetic moment.

are taken to be on the mass shell, $p^2 = p'^2 = m^2$, while the external photon line is in general off-shell. Within the covariant perturbation expansion we are using, the $\mathscr{M}_\mu$ should be a Lorentz four-vector, and it is a function of two independent four-momenta p and p'.

Thus, one may guess immediately that the most general form of the $\mathscr{M}_\mu$ could be

$$\mathscr{M}_\mu = \bar{u}(p')\big[A_1(q^2)\gamma_\mu + A_2(q^2)p_\mu + A_3(q^2)p'_\mu\big]u(p)\,. \tag{F.2}$$

As regards the invariant amplitudes (form factors) $A_j(q^2)$, $j = 1, 2, 3$, these might depend on p^2, p'^2 and the scalar product $p \cdot p'$. But $p^2 = p'^2 = m^2$, so there is just one independent kinematical invariant, e.g. $q^2 = (p' - p)^2 = 2m^2 - 2p \cdot p'$. A remark is perhaps in order here. When the loop integrations inside the blob in Fig. F.1 are carried out, one gets some Lorentz invariant form factors and the resulting $\Gamma_\mu(p', p)$ incorporates diverse products of γ-matrices, which enter the game either as γ_μ, or slashed combinations $\not{p}$ and $\not{p}'$. To simplify the products of γ-matrices, one can employ their basic anticommutation relations, which lead e.g. to $\not{p}\gamma_\mu = 2p_\mu - \gamma_\mu\not{p}$, $\not{p}\not{p}' = 2p \cdot p' - \not{p}'\not{p}$, etc. In this way, one may eventually encounter just a finite number of γ-matrix products, such as

$$\not{p}\gamma_\mu,\ p_\mu\not{p}',\ \not{p}\gamma_\mu\not{p}',\ \ldots \tag{F.3}$$

The readers are encouraged to activate their imagination and try to find all possible relevant γ-matrix products involved here. By the way, a detailed explicit evaluation of $\Gamma_\mu(p', p)$ at one-loop level, sketched in Chapter 50, is quite instructive in this context. Higher-order diagrams can of course produce long chains of γ-matrices, but one can always employ the anticommutation relations, and move the matrices inside the chain in such a way that one eventually gets $(\not p)^2 = p^2 = m^2$ and similarly for $(\not p')^2$. Moreover, one should put factors $\not p$ and $\not p'$ in the right order, such that $\not p'$ stands on the left and $\not p$ on the right; one may then utilize Dirac equations $\bar{u}(p')\not p' = m\,\bar{u}(p')$ and $\not p u(p) = m\,u(p)$. A typical example of the above-mentioned manipulations is as follows: one may get easily, upon appropriate anticommutations of γ-matrices,

$$\not p \gamma_\mu \not p' = 2p_\mu \not p' + 2p'_\mu \not p - 2p \cdot p' \gamma_\mu - \not p' \gamma_\mu \not p\,.$$

In such a way, one can justify the simple structure (F.2).

Well, after such a long explanatory comment, let us take the form (F.2) for granted and proceed further. For our purpose, it is more convenient to recast the expression in terms of the combinations $p'_\mu + p_\mu$ and $p'_\mu - p_\mu$, so that we write Eq. (F.2) in an equivalent form

$$\mathscr{M}_\mu = \bar{u}(p')\big[A(q^2)\gamma_\mu + B(q^2)(p'_\mu + p_\mu) + C(q^2)(p'_\mu - p_\mu)\big]u(p)\,. \tag{F.4}$$

Now we may use the Ward–Takahashi (WT) identity (see Chapter 42, the formula (42.20)), which in our present notation reads simply

$$q^\mu \mathscr{M}_\mu = 0\,. \tag{F.5}$$

Using the decomposition (F.4) and the identities $\bar{u}(p')\not q u(p) = 0$, $(p'_\mu + p_\mu)(p'^\mu - p^\mu) = 0$, the WT relation (F.5) is reduced to $q^2 C(q^2) = 0$, or,

$$C(q^2) = 0\,. \tag{F.6}$$

Thus, the form (F.4) becomes

$$\mathscr{M}_\mu = \bar{u}(p')\big[A(q^2)\gamma_\mu + B(q^2)(p'_\mu + p_\mu)\big]u(p)\,, \tag{F.7}$$

and with the help of the Gordon identity (50.25) this can be immediately rewritten as a combination of terms involving γ_μ and $\sigma_{\mu\nu}q^\nu$. The formula (50.15) is thereby proven.

Finally, let us remark that we have demonstrated the validity of the WT identity at the one-loop level, but in fact it is quite general, as a consequence of the gauge invariance of QED. A detailed discussion of this topic can be found e.g. in the book [6], Chapter 8, section 8.4.1. Thus, we may conclude that the form (50.15) is valid to any order of perturbation expansion.